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On Freud's Equations for Exponential Weights

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DEDICATED TO THE MEMORY OF GÉZA FREUD

Let $\{p_n\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials associated with the weight $\exp(-f(x))$, $x \in (-\infty, \infty)$, where f is a polynomial of even degree with positive leading coefficient. The coefficients of the three-term recurrence relation

$$a_{n+1} p_{n+1}(x) = (x - b_n) p_n(x) - a_n p_{n-1}(x),$$

are shown to be unique "admissible" solution of the equations

$$F_n(a, b) = 0, \qquad n = 1, 2, \ldots,$$

$$G_n(a, b) = 0, \qquad n = 0, 1\ 2, \ldots,$$

already considered by Freud for $f(x) = x^{2m}$. Using these equations, we prove an important special case of Freud's Conjecture. More precisely, we establish the asymptotic behaviour of $\{a_n\}$ and $\{b_n\}$ for the weight $\exp(-f(x))$. Further, we suggest extensions of the method used here, which should lead to a proof in the general case $f(x) = |x|^{\alpha}$, $\alpha > 1$. © 1986 Academic Press, Inc.

1. INTRODUCTION

The elements of description of the classical orthogonal polynomials are generating functions, differential equations, differential relations, Rodrigues formula, and explicit formulas for the coefficients in the three-term recurrence relation. Each of these elements have been extended, giving rise

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to families of nonclassical orthogonal polynomials (Pollaczek's generating function, functional relations and equations, discrete Rodrigues formula—see [14, Chap. 6; 19; 62; 73; 74].

This paper is concerned with a technique which produces usually implicit equations for the coefficients a_n and b_n in the three-term recurrence relation. Using these equations, we show how to deduce properties of $\{a_n\}$ and $\{b_n\}$, especially their asymptotic behavior.

To discuss the results, we need some notation. Throughout, let m be a positive integer, and let $f(x)$ be a polynomial of even degree, with positive leading coefficient, so that

$$f(x)=\sum_{i=0}^{2m} c_i x^{2m-i}, \qquad x\in R,\ c_0>0. \tag{1.1}$$

Further, let

$$w(x)=\exp(-f(x)), \qquad x\in R, \tag{1.2}$$

and let p_n, $n=0, 1, 2,...$, be the orthonormal polynomials with respect to w, so that

$$\int_{-\infty}^{\infty} p_n(x)\, p_k(x)\, w(x)\, dx=\begin{cases}0, & k\neq n.\\ 1, & k=n,\end{cases} \tag{1.3}$$

Let $\gamma_n>0$ denote the leading coefficient of p_n, $n=0, 1, 2,....$ The orthonormal polynomials satisfy the three term recurrence relation

$$a_{n+1}\, p_{n+1}(x)=(x-b_n)\, p_n(x)-a_n\, p_{n-1}(x), \qquad n=0, 1, 2,..., \tag{1.4}$$

with $a_0=0$ and $a_n=\gamma_{n-1}/\gamma_n$, $n=1, 2, 3,....$

A concise description of (1.4) uses the Jacobi matrix

$$A=\begin{bmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & a_2 & b_2 & a_3 \\ & & & \ddots \end{bmatrix} \tag{1.5}$$

and the column vector

$$\mathbf{p}=\mathbf{p}(x)=\begin{bmatrix} p_0(x)\\ p_1(x)\\ p_2(x)\\ \vdots\end{bmatrix}.$$

Clearly we may rewrite (1.3) in the form

$$x\mathbf{p} = A\mathbf{p}. \tag{1.6}$$

By repeatedly applying (1.6), we obtain

$$x^j\mathbf{p} = A^j\mathbf{p}, \qquad j = 0, 1, 2, \ldots,$$

and hence for any polynomial $Q(x)$,

$$Q(x)\,\mathbf{p} = Q(A)\,\mathbf{p}. \tag{1.7}$$

Given nonnegative integers j and k, $(\mathbf{p})_j$ will denote the element in the $(j+1)$th row of $\mathbf{p}$, while $(Q(A))_{j,k}$ will denote the element in the $(j+1)$th row and $(k+1)$th column of $Q(A)$. Using (1.7) and orthonormality, we see that

$$\begin{aligned}\int_{-\infty}^{\infty} Q(x)\,p_j(x)\,p_k(x)\,w(x)\,dx &= \int_{-\infty}^{\infty} (Q(x)\,\mathbf{p})_j\,p_k(x)\,w(x)\,dx \\ &= \int_{-\infty}^{\infty} (Q(A)\,\mathbf{p})_j\,p_k(x)\,w(x)\,dx. \\ &= (Q(A))_{j,k},\ j, k = 0, 1, 2, \ldots. \end{aligned} \tag{1.8}$$

Throughout, given sequences of real numbers $a = (a_1, a_2, a_3, \ldots)$ and $b = (b_0, b_1, b_2, \ldots)$, we define the associated Jacobi matrix A by (1.5) and can define

$$F_n(a, b) = a_n(f'(A))_{n,n-1}, \qquad n = 1, 2, 3, \ldots, \tag{1.9}$$

$$G_n(a, b) = f'(A))_{n,n}, \qquad n = 0, 1, 2, \ldots. \tag{1.10}$$

The equations on which we base our analysis are described by the following lemma:

LEMMA 1.1. *Let f and w be given by (1.1) and (1.2), respectively. Then the coefficients $\{a_n\}$ and $\{b_n\}$ in the recurrence relation (1.4) satisfy the equations*

$$F_n(a, b) = a_n(f'(A))_{n,n-1} = n, \qquad n = 1, 2, 3, \ldots, \tag{1.11}$$

$$G_n(a, b) = f'(A)_{n,n} = 0, \qquad n = 0, 1, 2, \ldots. \tag{1.12}$$

Proof. Let

$$I_{n,k} = \int_{-\infty}^{\infty} (p_n p_{n-k})'(x)\,w(x)\,dx, \qquad k = 0, n = 0, 1, 2, \ldots,$$

$$k = 1, n = 1, 2, 3, \ldots. \tag{1.13}$$

We see that, by orthogonality,

$$I_{n,0} = 2 \int_{-\infty}^{\infty} p_n(x)\, p_n'(x)\, w(x)\, dx = 0,$$

while integrating by parts,

$$I_{n,0} = p_n^2(x)\, w(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} p_n^2(x)\, w'(x)\, dx$$

$$= \int_{-\infty}^{\infty} p_n^2(x)\, f'(x)\, w(x)\, dx = (f'(A))_{n,n},$$

by (1.8). Thus (1.12) is valid. Next, by (1.13),

$$I_{n,1} = \int_{-\infty}^{\infty} p_n'(x)\, p_{n-1}(x)\, w(x)\, dx + \int_{-\infty}^{\infty} p_n(x)\, p_{n-1}'(x)\, w(x)\, dx$$

$$= n\gamma_n \int_{-\infty}^{\infty} (x^{n-1} + \cdots)\, p_{n-1}(x)\, w(x)\, dx + 0$$

(by orthogonality and where $\cdots$ is a polynomial of degree $\leqslant n-2$)

$$= n\gamma_n/\gamma_{n-1} = n/a_n, \tag{1.14}$$

by (1.3) and (1.4). Next, integrating by parts, as above, we see that

$$I_{n,1} = \int_{-\infty}^{\infty} p_n(x)\, p_{n-1}(x)\, f'(x)\, w(x)\, dx = (f'(A))_{n,n-1}, \tag{1.15}$$

by (1.8). Thus (1.11) also follows. ∎

In this paper, we shall call (1.11) and (1.12) Freud's equations, although they can be traced back to Laguerre [24] (see [38] for additional references). However, Freud [16–18] initiated investigation of the properties of the solutions of (1.11) and (1.12). We remark that it is possible to define analogues of (1.11) and (1.12) even when $w'(x)/w(x)$ is a rational function; see Shohat [72].

Some facts on the algebra of tridiagonal matrices and their consequences on the form of the Freud's equations are gathered in the two following lemmas.

LEMMA 1.2. *Let A be the Jacobi matrix* (1.5). *Then*

(i) *The matrix A^k is a symmetric band matrix of bandwidth $2k+1$,*

$$(A^k)_{n,n+j} = 0, \qquad \text{if} \quad |j| > k.$$

(ii) *The extreme elements of the* $(n+1)$th *row of* A^k *are*

$$(A^k)_{n,n-k} = a_n a_{n-1} \cdots a_{n-k+1}, \qquad n = k, k+1, \ldots,$$
$$(A^k)_{n,n+k} = a_{n+1} a_{n+2} \cdots a_{n+k}, \qquad n = 0, 1, \ldots.$$

(iii)

$$a_n a_{n-1} \cdots a_{n-j+1} \quad \text{is a factor of } (A^k)_{n,n-j} \qquad 1 \leqslant j \leqslant k,$$
$$a_{n+1} a_{n+2} \cdots a_{n+j} \quad \text{is a factor of } (A^k)_{n,n+j} \qquad 1 \leqslant j \leqslant k.$$

(iv) *Let*

$$x_{2i} = a_i, \qquad i = 1, 2, 3, \ldots, \tag{1.16}$$
$$x_{2i+1} = b_i, \qquad i = 0, 1, 2, \ldots, \tag{1.17}$$

then $(A^k)_{n,n+j}$ *is a homogeneous polynomial of degree k in the variables*

$$x_{2n+2-k+j}, x_{2n+3-k+j}, \ldots, x_{2n+k+j-1}, x_{2n+k+j}, \qquad -k \leqslant j \leqslant k.$$

Proof. These properties are established by induction on k. The sum

$$(A^k)_{n,n+j} = \sum_{i=-1}^{1} (A)_{n,n+i} (A^{k-1})_{n+i,n+j} \tag{1.18}$$

contains at most three terms. If (i) is true for $k-1$, it holds for k, as $|j| > k$ implies $|j-i| > k-1$. If $|j| = k$, $(A^k)_{n,n+j}$ reduces to a single term $a_n(A^{k-1})_{n-1,n-k}$ if $j = -k$, $a_{n+1}(A^{k-1})_{n+1,n+k}$ if $j = k$ and (ii) follows. If the first part of (iii) is true for $(A^{k-1})_{n,n-j}$, $j \geqslant 1$, then the two last terms of

$$(A^k)_{n,n-j} = a_n(A^{k-1})_{n-1,n-j} + b_n(A^{k-1})_{n,n-j} + a_{n+1}(A^{k-1})_{n+1,n-j}$$

contain the required factor, and so does the first one, thanks to the a_n factor, even if $j = 1$ $(n-j = n-1-(j-1))$. The second part follows from the symmetry of A^k.

For the proof of (iv), (1.18) is written

$$(A^k)_{n,n+j} = \sum_{i=-1}^{1} x_{2n+i+1} (A^{k-1})_{n+i,n+j}.$$

If (iv) is true for $k-1$, $(k \geqslant 2)$ each term depends on

$$x_{2n+i+1} \quad \text{and} \quad x_{2n+i-k+j+3}, \ldots, \quad x_{2n+i+k+j-1}$$

provided $|j-i| \leqslant k-1$ (otherwise, the term vanishes). We want to show that these variables are indeed in the set

$$\{x_{2n+2-k+j}, \ldots, x_{2n+k+j}\},$$

i.e.,

$$2n+2-k+j \leqslant 2n+i+1 \leqslant 2n+k+j,$$
$$2n+2-k+j \leqslant 2n+i-k+j+3 \leqslant 2n+i+k+j-1 \leqslant 2n+k+j,$$

if $|i| \leqslant 1$ and $|j-i| \leqslant k-1$. This is readily checked: the last inequality implies

$$j+k \geqslant i+1 \qquad \text{and} \qquad j-k \leqslant i-1. \quad \blacksquare$$

LEMMA 1.3. *With f given by* (1.1), *the general form of $F_n(a, b)$ and $G_n(a, b)$ is*

$$\begin{aligned} F_n(a, b) &= a_n^2 \Bigg[\sum_{i=0}^{m-1} c_{2i} P_{2i}(a_{n-m+1+i},\dots, a_{n+m-1-i}; b_{n-m+1+i}, b_{n+m-2-i}) \\ &\quad + c_{2i+1} P_{2i+1}(a_{n-m+2+i},\dots, a_{n+m-2-i}; b_{n-m+1+i},\dots, b_{n+m-2-i}) \Bigg], \end{aligned} \tag{1.19}$$

$$\begin{aligned} G_n(a, b) &= \sum_{i=0}^{m-1} c_{2i} Q_{2i}(a_{n-m+2+i},\dots, a_{n+m-1-i}; b_{n-m+1+i},\dots, b_{n+m-1-i}) \\ &\quad + c_{2i+1} Q_{2i+1}(a_{n-m+2+i},\dots, a_{n+m-1-i}; b_{n-m+2+i},\dots, b_{n+m-2-i}), \end{aligned} \tag{1.20}$$

where P_i and Q_i are homogeneous polynomials of respective degrees $2m-i-2$ ($P_{2m-1} \equiv 0$) and $2m-i-1$.

Proof. We just have to apply Lemma 1.2, especially (iv) to (1.11) and (1.12), where

$$f'(A) = \sum_{i=0}^{2m-1} c_i(2m-i)\, A^{2m-i-1}.$$

Indeed, from Lemma 1.2(iv), $(A^{2m-i-1})_{n,n-1}$ depends on $x_{2n+i-2m+2},\dots, x_{2n-i+2m-2}$, and $(A^{2m-i-1})_{n,n}$ depends on $x_{2n+i-2m+3},\dots, x_{2n+2m-i-1}$. Equations (1.19) and (1.20) follow then from (1.16) and (1.17). From Lemma 1.2(iii), a_n is a factor of $(A^{2m-i-1})_{n,n-1}$, $i=0,\dots, 2m-2$, and therefore a_n^2 is a factor of $F_n(a, b)$. $\blacksquare$

At this stage, a simple example facilitates understanding and allows to check the preceding lemmas.

EXAMPLE 1.4. (cf. Bauldry [4]). Let $m=2$. Then (1.11) and (1.12) take the form

$$F_n(a, b)=a_n^2[4c_0(a_{n-1}^2+b_{n-1}^2+b_{n-1}b_n+a_n^2+b_n^2+a_{n+1}^2)$$
$$+3c_1(b_{n-1}+b_n)+2c_2]=n,$$
$$n=1, 2, 3,..., \qquad (1.21)$$

$$G_n(a, b)=4c_0(a_n^2 b_{n-1}+2a_n^2 b_n+b_n^3+2a_{n+1}^2 b_n+a_{n+1}^2 b_{n+1})$$
$$+3c_1(a_n^2+b_n^2+a_{n+1}^2)+2c_2 b_n+c_3=0,$$
$$n=0, 1,.... \qquad (1.22)$$

Recall that $a_0=0$.

We shall use the usual $\circ$, 0, and $\sim$ notation. Thus, for example, $c_n \sim d_n$ if for large enough n, the ratio c_n/d_n is bounded above and below by positive constants independent of n. Throughout $C, C_1, C_2, C_3,...,$ denote positive constants independent of n.

The paper is organized as follows: In Section 2, we investigate unicity of the solutions of (1.11) and (1.12). In Section 3, we find bounds for the solutions, and in Section 4, we investigate the asymptotic behaviour of certain approximation solutions. In Section 5, we estimate a certain Fréchet derivative associated with (F_n, G_n), $n=1, 2,...,$ and in Section 6, we prove our main result, Theorem 6.1, which proves the Freud's Conjecture (cf. [16–18]) for $w(x)=\exp(-f(x))$ given by (1.1) and (1.2), and we also give an estimate for the remainder. Finally, in Section 7, we outline steps which should lead to a proof of Freud's Conjecture for $w(x)=\exp(-|x|^\alpha)$, all $\alpha>1$.

It is noteworthy that the results of this paper will have several applications. For the special case $g(x)=x^{2m}$, Máté, Nevai, and Zaslavsky [43] used the results of [38] and [40] to obtain an asymptotic expansion for a_n(cf. [5]), and Nevai [55] used these asymptotics to find sharp bounds for the corresponding orthogonal polynmials (cf. [10] and [31]). Asymptotic expansions for the recurrence coefficients are essential ingredients for investigating properties of the zeros of the orthogonal polynomials (cf. [17, 18, 41, 42, 44, 52, 55, 56, 64–66, 75–78]), and for finding various asymptotics for the orthogonal polynomials, a program initiated by P. Nevai and his students (cf. [4, 52–56, 67, 68]). We refer to Nevai's survey [57] for additional information.

2. UNICITY OF ADMISSIBLE SOLUTIONS

In this section, we shall prove that Eqs (1.11) and (1.12) have a unique "admissible" solution. First, however, we need a definition of admissibility:

DEFINITION 2.1. Let f and w be given by (1.1) and (1.2), respectively. We say that real numbers $a=(a_1, a_2, a_3,...)$ and $b=(b_0, b_1, b_2,...)$ form an *admissible solution* of (1.11) and (1.12) if they satisfy (1.11) and (1.12) and if $a_n>0$, $n=1, 2, 3,...$.

We shall prove

THEOREM 2.2. *Let f and w be given by* (1.1) *and* (1.2), *respectively. Then* (1.11) *and* (1.12) *have a unique admissible solution.*

First, however, we need

LEMMA 2.3. *Let f and w be given by* (1.1) *and* (1.2), *respectively. Let $\beta(t)$, $t\in(-\infty, \infty)$, be a monotone increasing function such that all moments of $d\beta$ are finite, and assume that*

$$\int_{-\infty}^{\infty} P(x)\, f'(x)\, d\beta(x)=\int_{-\infty}^{\infty} P'(x)\, d\beta(x), \tag{2.1}$$

for all polynomials P. Further assume that β is normalized so that $\beta(-\infty)=0$ and

$$\int_{-\infty}^{\infty} d\beta(x)=\int_{-\infty}^{\infty} w(x)\, dx. \tag{2.2}$$

Then β is absolutely continuous in $(-\infty, \infty)$, and

$$\beta'(x)=w(x), \qquad x\in(-\infty, \infty). \tag{2.3}$$

Proof. We shall use ideas from [11] and [12], in a suitably modified form. Let

$$\mu_n=\int_{-\infty}^{\infty} x^n d\beta(x), \qquad n=0, 1, 2,....$$

Step 1. *Estimation of μ_n.* We have, by (2.1),

$$\int_{-\infty}^{\infty} x^{n+1} f'(x)\, d\beta(x)=(n+1)\int_{-\infty}^{\infty} x^n d\beta(x), \qquad n=0, 1, 2,...,$$

which implies

$$\int_{-\infty}^{\infty} x^n[xf'(x)-(n+1)]\, d\beta(x)=0, \qquad n=0, 1, 2,.... \tag{2.4}$$

Now, by (1.1), there exists $C_1>0$, C_2, C_3, C_4 such that

$$|xf'(x)|\leqslant C_2x^{2m}+C_3, \quad x\in R; \qquad xf'(x)\geqslant C_1x^{2m}, \quad |x|\geqslant C_4. \tag{2.5}$$

Let

$$\chi_n = ((n+2)/C_1)^{1/(2m)}, \qquad n = 1, 2, 3, \dots \tag{2.6}$$

For n large enough (2.5) shows that if $|x| \geqslant \chi_n$, then

$$xf'(x) - (n+1) \geqslant C_1(n+2)/C_1 - (n+1) = 1.$$

Now we are able to find a bound for μ_n when n is even and large enough:

$$\mu_n = \int_{|x| \leqslant \chi_n} x^n d\beta(x) + \int_{|x| \geqslant \chi_n} x^n d\beta(x).$$

The first integral is of course bounded by $(\chi_n)^n \int_{-\infty}^{\infty} d\beta(x)$. The second one, as $1 \leqslant xf'(x) - (n+1)$, is bounded by

$$\begin{aligned}
&\int_{|x| \geqslant \chi_n} x^n [xf'(x) - (n-1)]\, d\beta(x) \\
&\quad = -\int_{|x| < \chi_n} x^n [xf'(x) - (n+1)]\, d\beta(x) \qquad \text{(by (2.4))} \\
&\quad \leqslant \int_{|x| < \chi_n} x^n [C_2 x^{2m} + C_3 + n + 1]\, d\beta(x) \qquad \text{(by (2.5))} \\
&\quad \leqslant (\chi_n)^n [C_2(\chi_n)^{2m} + C_3 + n + 1] \int_{-\infty}^{\infty} d\beta(x).
\end{aligned}$$

The definition (2.6) of χ_n shows thât a bound

$$\mu_n \leqslant (Cn)^{n/(2m)} \tag{2.7}$$

holds for n even and large enough.

Step 2. *Fourier transform identity.* We shall use (2.1) and (2.7) to show that for each fixed real t,

$$\int_{-\infty}^{\infty} e^{itx} f'(x)\, d\beta(x) = it \int_{-\infty}^{\infty} e^{itx} d\beta(x). \tag{2.8}$$

Note first, that (2.8) is true if we replace e^{itx} by its partial sums. More precisely, by (2.1), for $n = 1, 2, 3, \dots$,

$$\int_{-\infty}^{\infty} \left\{ \sum_{j=0}^{n} (itx)^j / j! \right\} f'(x)\, d\beta(x) = it \int_{-\infty}^{\infty} \left\{ \sum_{j=0}^{n-1} (itx)^j / j! \right\} d\beta(x). \tag{2.9}$$

To prove (2.8) from (2.9), we must estimate

$$T_1(n)=\int_{-\infty}^{\infty}\left|e^{itx}-\sum_{j=0}^{n}(itx)^j/j!\right|\,|f'(x)|\,d\beta(x)$$

and

$$T_2(n)=\int_{-\infty}^{\infty}\left|e^{itx}-\sum_{j=0}^{n-1}(itx)^j/j!\right|\,d\beta(x).$$

By applying Taylor's formula to the real and imaginary parts of e^{iu}, we deduce that for $n=1, 2, 3,...,$

$$\left|e^{iu}-\sum_{j=0}^{n-1}(iu)^j/j!\right|\leqslant 2|u|^n/n!, \qquad u\in(-\infty,\infty) \tag{2.10}$$

(cf. Freud [15, p. 79]). Hence if n is even,

$$\begin{aligned} T_1(n) &\leqslant 2|t|^{n+1}\int_{-\infty}^{\infty}|x|^{n+1}|f'(x)|\,d\beta(x)/(n+1)! \\ &\leqslant 2|t|^{n+1}\{C_2\mu_{n+2m}+C_3\mu_n\}/(n+1)! \qquad \text{(by (2.5))} \\ &= 0(|t|^{n+1}n^{(n+2m)/(2m)}/n!)\to 0 \text{ as } n\to\infty \qquad \text{(by (2.7))} \end{aligned}$$

since $2m>1$. In a similar, but easier, manner, we may use (2.10) to show that $T_2(n)\to 0$, $n\to\infty$, n even. Clearly then (2.8) now follows.

Step 3. *Completion of the proof.* First, letting $t=0$ in (2.8), we obtain

$$\int_{-\infty}^{\infty} f'(x)\,d\beta(x)=0. \tag{2.11}$$

Further, letting $t\to\infty$ in (2.8), we see that

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} e^{itx}d\beta(x)=0.$$

It then follows from Theorem 4.19 in Zygmund [81, Vol. 2, pp. 258–261] that

$$\beta(y)=\int_{-\infty}^{y} d\beta(x)$$

is continuous in R. Integrating the left member of (2.8) by parts, cancelling it, and using (2.11), we see that

$$\int_{-\infty}^{\infty} e^{itx}\left\{\int_{-\infty}^{x}(-f'(u))\,d\beta(u)\right\}dx=\int_{-\infty}^{\infty} e^{itx}d\beta(x), \tag{2.12}$$

for t real, $t \neq 0$. We may deduce that (2.12) still holds for $t=0$ by using Lebesgue's Dominated Convergence Theorem, (2.11) and the fact that all moments of $d\beta$ are finite, while $|f'|$ is of polynomial growth. We may now use uniqueness of Fourier transforms (Zygmund [81, Vol. 2, Theorem 10.15, p. 293]) to deduce that

$$\beta(y)=\int_{-\infty}^{y}\int_{-\infty}^{x}(-f'(u))\,d\beta(u)\,dx, \qquad y\in(-\infty,\infty).$$

It follows that β is absolutely continuous and

$$\beta'(x)=-\int_{-\infty}^{x} f'(u)\,\beta'(u)\,du, \qquad x\in(-\infty,\infty),$$

so that β' is also absolutely continuous and

$$\beta''(x)=-f'(x)\,\beta'(x), \qquad x\in(-\infty,\infty).$$

Integrating, we obtain for some $K>0$,

$$\beta'(x)=K\exp(-f(x)), \qquad x\in(-\infty,\infty).$$

The normalization condition (2.3) ensures that $K=1$. ∎

Remark 2.4. Note that the above lemma is still valid if f' is bounded in each finite interval and if for some $\eta>0$,

$$f'(x)\sim \operatorname{sign}(x)|x|^{\eta}, \qquad |x| \text{ large enough},$$

even if f is not a polynomial. One needs only to replace $2m$ by $1+\eta$ in Step 1 of the lemma. This gives

$$\mu_n \leqslant (Cn)^{n/(1+\eta)}, \qquad n \quad \text{even and positive.}$$

instead of (2.7). Then, for the purposes of Step 2 (the Fourier transform identity),

$$\int_{-\infty}^{\infty}|x|^{n+1}|f'(x)|\,d\beta(x) \leqslant \int_{-\infty}^{\infty}[C_2|x|^{n+1+\eta}+C_3|x|^{n}]\,d\beta(x)$$
$$\leqslant C_5+(C_2+C_3)\,\mu_{n+k},$$

where $n+k$ is the smallest even integer $\geqslant n+1+\eta$.

Proof of Theorem 2.2. We know from Lemma 1.1 that there is at least one admissible solution. Let us now assume that a_n^*, $n=1, 2, 3,\ldots,$ and b_n^*,

$n=0, 1, 2,\ldots$, are an admissible solution. Let us define polynomials p_n^* of degree n, with leading coefficient γ_n^*, by $p_0^*(x)=1$, and

$$a_{n+1}^* p_{n+1}^*(x)=(x-b_n^*)\, p_n^*(x)-a_n^*\, p_{n-1}^*(x), \qquad n=1, 2, 3,\ldots. \tag{2.13}$$

As is well known (Freud [15, Theorem 2.1.5, p. 60]), there is then a nondecreasing function $\beta(x)$, $x\in(-\infty, \infty)$, such that p_n^*, $n=0, 1, 2,\ldots$, are orthogonal with respect to $d\beta$. We can normalize β so that $\beta(-\infty)=0$ and so that β satisfies (2.2). Next, note that by (2.13) and orthogonality

$$\begin{aligned}\int_{-\infty}^{\infty} &(a_{n+1}^* p_{n+1}^*(x))\, p_{n+1}^*(x)\, d\beta(x)\\ &=\int_{-\infty}^{\infty} xp_n^*(x)\, p_{n+1}^*(x)\, d\beta(x)\\ &=\int_{-\infty}^{\infty} p_n^*(x)\{a_{n+2}^* p_{n+2}^*(x)+b_{n+1}^* p_{n+1}^*(x)\\ &\quad +a_{n+1}^* p_n^*(x)\}\, d\beta(x) \quad \text{(by 2.13))}\\ &=\int_{-\infty}^{\infty} a_{n+1}^*(p_n^*(x))^2 d\beta(x).\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} (p_{n+1}^*(x))^2 d\beta(x)=\int_{-\infty}^{\infty} (p_n^*(x))^2 d\beta(x), \qquad n=0, 1, 2,\ldots.$$

Hence by changing p_0^*, we may assume that p_n^*, $n=0, 1, 2,\ldots$, are orthonormal with respect to $d\beta$. Now let A^* be the Jacobi matrix associated with $\{a_n^*\}$ and $\{b_n^*\}$. Exactly as at (1.8), we see that for $k=0$ and $n=0, 1, 2,\ldots$, and for $k=1$ and $n=1, 2, 3,\ldots$,

$$\int_{-\infty}^{\infty} f'(x)\, p_n^*(x)\, p_{n-k}^*(x)\, d\beta(x)=(f'(A^*))_{n,n-k} =\begin{cases}0, & k=0\\ n/a_n^*, & k=1\end{cases}$$

by (1.11) and (1.12). Further, exactly as in the proof of Lemma 1.1, we see that since $a_n^*=\gamma_{n-1}^*/\gamma_n^*$,

$$\int_{-\infty}^{\infty} (p_n^*(x)\, p_{n+k}^*(x))' d\beta(x)=\begin{cases}0, & k=0\\ n/a_n^*, & k=1\end{cases}$$

Hence for $k=0$ and $n=0, 1, 2, \ldots$, and for $k=1$ and $n=1, 2, 3, \ldots$,

$$\int_{-\infty}^{\infty} (p_n^* p_{n-k}^*)'(x)\, d\beta(x) = \int_{-\infty}^{\infty} f'(x)(p_n^* p_{n-k}^*)(x)\, d\beta(x).$$

Since $\{p_n^{*2}\}$ and $\{p_n^* p_{n-1}^*\}$ span the polynomials, we see that (2.1) is true. Hence (2.3) follows. Finally, as the weight uniquely defines the coefficients in the recurrence relation, the proof is complete. ∎

3. Bounds

Knowledge on the rate of growth of a_n and b_n is important for further work. We prove

Theorem 3.1. *Let f and w be given by* (1.1) *and* (1.2), *respectively. Then the admissible solution $\{a_n\}$ and $\{b_n\}$ of* (1.11) *and* (1.12) *satisfies*

$$a_n \sim n^{1/(2m)}, \qquad n=1, 2, 3, \ldots \tag{3.1}$$

and

$$|b_n| = O(n^{1/(2m)}), \qquad n=1, 2, 3, \ldots. \tag{3.2}$$

Proof. First, we note the following inequality: There exists constants C and C_1 such that for every polynomial P of degree at most n,

$$\int_{-\infty}^{\infty} |P(x)|\, w(x)\, dx \leqslant C_1 \int_{-Cn^{1/(2m)}}^{Cn^{1/(2m)}} |P(x)|\, w(x)\, dx. \tag{3.3}$$

See [30, Lemma 6.1] for a proof of (3.3). Note that in [30]. $n^{1/(2m)}$ is replaced by q_n, which for large enough n, is the positive root of the equation

$$q_n f'(q_n) = n.$$

Clearly $q_n = \{n/(2mc_0)\}^{1/(2m)}(1 + \circ(1))$, $n \to \infty$, and so (3.3) follows. Next, it is an easy consequence of (1.4) and orthonormality that

$$\begin{aligned} a_n^2 + b_n^2 + a_{n+1}^2 &= \int_{-\infty}^{\infty} x^2 (p_n(x))^2 w(x)\, dx \\ &\leqslant C_1 \int_{-C(2n+2)^{1/(2m)}}^{C(2n+2)^{1/(2m)}} x^2 (p_n(x))^2 w(x)\, dx \qquad \text{(by (3.3))} \\ &\leqslant C_1 n^{1/m}, \end{aligned}$$

by orthonormality. This yields the upper bounds in (3.1) and (3.2).

Lower bounds for a_n are still not so frequent in the literature. We may remark that the equivalent of (3.1) for $f(x)=|x|^\alpha$ (α real $\geqslant 1$) has already been established by Nevai [49], and that a general result on this matter can be found in Knopfmacher's work [23]. Here, we shall obtain these lower bounds by using (1.14) and (1.15), which show that

$$\begin{aligned} n/a_n &= \int_{-\infty}^{\infty} p_n(x)\, p_{n-1}(x)\, f'(x)\, w(x)\, dx \\ &\leqslant C_1 \int_{-C(2n+m)^{1/(2m)}}^{C(2n+m)^{1/(2m)}} |p_n(x)\, p_{n-1}(x)|\, |f'(x)|\, w(x)\, dx \qquad \text{(by (3.3))} \\ &\leqslant C_2 n^{1-1/(2m)}, \end{aligned}$$

by the Cauchy–Schwarz inequality, orthonormality, and as $|f'(x)| = O(|x|^{2m-1})$, $|x| \to \infty$. ▮

4. Consistent Approximate Solution

The expansion of (1.11) and (1.12) as polynomials in $\{a_n\}$ and $\{b_n\}$ allow some guesses on the asymptotic behaviour of the solution. For instance,

$$a_n \sim (n/12c_0)^{1/4} \qquad \text{and} \qquad b_n \sim 0$$

solve approximately (1.21) and (1.22). This is an example of consistent approximate solution:

Definition 4.1. Let f and w be given by (1.1) and (1.2), respectively. We say that the real numbers $a' = (a'_1, a'_2, a'_3, ...)$ and $b' = (b'_0, b'_1, b'_2, ...)$ form a *consistent approximate solution* of (1.11) and (1.12) if

$$\begin{aligned} F_n(a', b') - n &= \circ(n), & n \to \infty \\ G_n(a', b') &= \circ(n^{1-1/2m}), & n \to \infty. \end{aligned} \tag{4.1}$$

The necessity of the small o conditions is explained by the remark that any $\{a'_n, b'_n\}$ satisfying the upper bounds in (3.1) will satisfy (4.1), but with small o replaced by large O. Hence (4.1) is the very least we may ask for an approximate solution to be useful. A simple consistent approximate solution ([16, 38]) is

$$a'_n = [n/(c_0 C(2m))]^{1/2m} \qquad \text{and} \qquad b'_n = 0, \qquad n = 0, 1, 2, ...,$$

where

$$C(2m) = 2\Gamma(2m)/(\Gamma(m))^2 = 2(2m-1)!/((m-1)!)^2. \tag{4.2}$$

This is obtained by considering only the c_0x^{2m} term of f. As the neglected terms have degrees $2m-1$ and $2m-2$ in the a's and the b's, (Lemma 1.3), the remainders in (4.1) are $O(n^{1-1/(2m)})$ and $O(n^{1-2/(2m)})$.

However, a better approximation is achieved if one solves (1.11) and (1.12) separately for each n, assuming

$$a'_{n+k}=a'_n=\alpha, \qquad b'_{n+k}=b'_n=\beta, \qquad k=\pm 1, \pm 2, ..., \pm(m-1).$$

This leaves two nonlinear equations

$$\tilde{F}(a'_n, b'_n)=n \qquad \text{and} \qquad \tilde{G}(a'_n, b'_n)=0, \tag{4.3}$$

in two unknowns a'_n and b'_n.

For instance, if $m=2$ (cf. [4]), from (1.21) and (1.22),

$$\tilde{F}(a'_n, b'_n)=a'^2_n[12c_0(a'^2_n+b'^2_n)+6c_1b'_n+2c_2]=n,$$

$$\tilde{G}(a'_n, b'_n)=4c_0(6a'^2_nb'_n+b'^3_n)+3c_1(2a'^2_n+b'^2_n)+2c_2b'_n+c_3=0.$$

$\tilde{F}$ and $\tilde{G}$ may be described in the following way: Let $\tilde{A}$ be the (doubly infinite) Toeplitz matrix

$$\tilde{A}=\begin{bmatrix} \ddots & & & & \\ \alpha & \beta & \alpha & & \\ & \alpha & \beta & \alpha & \\ & & & & \ddots \end{bmatrix} \tag{4.4}$$

having symbol $\alpha z^{-1}+\beta+\alpha z$. Then $f'(A)$ is itself a Toeplitz matrix of symbol $f'(\alpha z^{-1}+\beta+\alpha z)$. Further, (1.11) and (1.12) become the equations (4.3), involving the coefficients of z^{-1} and z^0 in the Laurent expansion of $f'(\alpha z^{-1}+\beta+\alpha z)$ [21]:

$$\tilde{F}(\alpha, \beta)=\alpha(f'(\alpha z^{-1}+\beta+\alpha z))_{-1} \tag{4.5}$$

and

$$\tilde{G}(\alpha, \beta)=(f'(\alpha z^{-1}+\beta+\alpha z))_0. \tag{4.6}$$

An interesting alternative form for $\tilde{F}(\alpha, \beta)$ and $\tilde{G}(\alpha, \beta)$ is as Fourier coefficients: We see that

$$\tilde{F}(\alpha, \beta)=(\alpha/\pi)\int_0^{\pi} f'(\beta+2\alpha\cos\theta)\cos\theta\, d\theta$$

and

$$\tilde{G}(\alpha, \beta)=(1/\pi)\int_0^{\pi} f'(\beta+2\alpha\cos\alpha)\, d\theta,$$

showing that $n-\tilde{F}$ and $-\tilde{G}/2$ are the partial derivatives in $\log\alpha$ and β of the *Mhaskar and Saff's function*

$$M_n(\alpha, \beta) = n \log \alpha - (2\pi)^{-1} \int_0^\pi f(\beta + 2\alpha \cos \theta)\, d\theta. \tag{4.7}$$

Mhaskar and Saff [46, 47] showed that nth degree polynomials achieving least L_∞ norm (weighted by $(w(x))^{1/2} = \exp(-f(x)/2)$) must be considered only in the interval $[b'_n - 2a'_n,\ b'_n + 2a'_n]$, where a'_n and b'_n maximize $M_n(\alpha, \beta)$. It is not surprising to find here that they are also relevant to the L_2 norm extremal problem (connections between different norms have been worked in [44]) (cf. [66]). We are going to prove that the solution of this maximization problem gives good consistent approximate solutions.

THEOREM 4.2. *Let f and w be given by* (1.1) *and* (1.2), *respectively. Then one can find a consistent approximate solution $\{a'_n, b'_n\}$ of* (1.11) *and* (1.12) *with $a'_n > 0$, such that as $n \to \infty$*

$$a'_n = \{n/(c_0 C(2m))\}^{1/(2m)}(1 + \circ(1)), \tag{4.8}$$

$$b'_n = \circ(n^{1/(2m)}), \tag{4.9}$$

$$F_n(a', b') - n = O(1), \tag{4.10}$$

$$G_n(a', b') = O(n^{-1/(2m)}). \tag{4.11}$$

The result of the maximization is described in

LEMMA 4.3. *For large n, $M_n(\alpha, \beta)$ is maximized on $E = (0, \infty) \times (-\infty, \infty)$ at*

$$\alpha = n^{1/(2m)} s_1(n^{-1m}) \quad \text{and} \quad \beta = s_2(n^{-1/m}), \tag{4.12}$$

where s_1 and s_2 are convergent power series whose first terms are, respectively, $(c_0 C(2m))^{-1/(2m)}$ and $-c_1/(2mc_0)$.

Proof. From (1.1) and (4.7), we have

$$M_n(\alpha, \beta) = n \log \alpha - \sum_{i=0}^{2m} c_i \sum_{0 \leqslant k \leqslant m - i/2} \binom{2m-i}{2k} \times D_k \alpha^{2k} \beta^{2m-i-2k}$$

$$= n \log \alpha - c_0 D_m \alpha^{2m} - [m(2m-1)\, c_0 \beta^2 + (2m-1)\, c_1 \beta + c_2] \times D_{m-1} \alpha^{2m-2} - \cdots,$$

where

$$D_k=(2\pi)^{-1}\int_0^{\pi}(2\cos\theta)^{2k}d\theta=\frac{(2k)!}{2(k!)^2};$$

remark, from (4.2), that $C(2k)=2kD_k$, $k=0, 1,...$.

Existence of a maximum of $M_n(\alpha, \beta)$ in E: let us show that $M_n(\alpha, \beta)\to -\infty$ when (α, β) tends towards the boundary of E. This holds obviously when $\alpha\to 0$. Next, if $\alpha^2+\beta^2$ is large, an upper bound of the form

$$M_n(\alpha, \beta)\leqslant n\log\alpha-c_0'(\alpha^2+\beta^2)^{2m}+c'', \qquad c_0'>0$$

is readily established and shows that M_n has a maximum in E. This maximum is of course reached at a point (α, β) where the partial derivatives $n-\tilde{F}$ and $\tilde{G}$ vanish. Remark that $M_n(\alpha, \beta)\to -\infty$ when $\alpha^2+\beta^2\to\infty$ holds for a very large class of functions f: one has just to ask $f(x)/\log|x|\to\infty$ when $x\to\pm\infty$. This condition is indeed considered by Mhaskar and Saff [46, 47].

At the maximum of $M_n(\alpha, \beta)$, $\alpha\to\infty$. Indeed, if α remains bounded, $M_n(\alpha, \beta)$ remains less than $C_1 n$ (and could become strongly negative, if β is large), whereas the choice $\alpha=C_2 n^{1/(2m)}$, $\beta=0$ gives already $M_n(\alpha, \beta)\geqslant C_3 n\log n$ for large n, $C_3>0$.

We must have $\beta=\circ(\alpha)$: if $|\beta|$ remains larger than some product $C_4\alpha$, $C_4>0$, when $\alpha\to\infty$,

$$\partial M_n(\alpha, \beta)/\partial\beta=-c_0\sum_{k=0}^{m-1}D_k'\alpha^{2k}\beta^{2m-2k-1}+O(\beta^{2m-2}), \qquad D_k'>0$$

will not vanish.

At the maximum, α must be $\sim n^{1/(2m)}$:

$$\alpha\partial M_n(\alpha, \beta)/\partial\alpha=n-c_0C(2m)\,\alpha^{2m}+\circ(\alpha^{2m})$$

will not vanish otherwise.

Series expansions: let $\bar{\alpha}=n^{-1/(2m)}\alpha$ and $x=n^{-1/m}$. Then,

$$n^{-1}\tilde{F}(\alpha, \beta)-1=c_0C(2m)\bar{\alpha}^{2m}-1+O(x)$$

and

$$n^{-1+1/m}\tilde{G}(\alpha, \beta)=2\bar{\alpha}^{2m-2}D_{m-1}(2mc_0\beta+c_1)+O(x)$$

are polynomials in x. At $x=0$, these two expressions are equal to zero if $\bar{\alpha}=(c_0C(2m))^{-1/(2m)}$ and $\beta=-c_1/(2mc_0)$. As the Jacobian matrix is not singular at this point, the series expansions follow. ∎

Proof of Theorem 4.2. Let a'_n and b'_n maximize $M_n(\alpha, \beta)$ on E, as discussed in the lemma. $F_n(a', b')$ and $G_n(a', b')$ are polynomials in $a'_{n-m+1},...,$ a'_{n+m-1}, $b'_{n-m+1},..., b'_{n+m-1}$ (see Lemma 1.3). From the smooth variation (4.12) of a'_n and b'_n with respect to n,

$$a'_{n+j} = a'_n + O(n^{1/(2m)-1}) = a'_n(1 + O(n^{-1})), \tag{4.13}$$

$$b'_{n+j} - b'_n = O(n^{-1}). \tag{4.14}$$

Each term of F_n and G_n is a product of k factors (at most $2m$ for F_n and $2m-1$ for G_n) of the form

$$\Pi = a'_{n+s_1} \cdots a'_{n+s_i} b'_{n+t_1} \cdots b'_{n+t_j},$$

where $i + j = k$. Since all $|s_p|$, $|t_p| \leqslant m-1$, (4.13) and (4.14) show that

$$\begin{aligned}\Pi &= (a'_n)^i (1 + O(n^{-1}))[(b'_n)^j + O(n^{-1})] \\ &= (a'_n)^i (b'_n)^j + O(n^{i/(2m)-1}),\end{aligned}$$

where $a'_n \sim n^{1/(2m)}$ and $b'_n = O(1)$ have been used, from (4.12). Reconstruction of $F_n(a', b')$ and $G_n(a', b')$ yields therefore

$$F_n(a', b') = \tilde{F}(a'_n, b'_n) + O(1) = n + O(1) \qquad \text{as} \quad i \leqslant k \leqslant 2m,$$

$$G_n(a', b') = \tilde{G}(a'_n, b'_n) + O(n^{-1/(2m)}) = O(n^{-1/(2m)}) \qquad \text{as} \quad i \leqslant k \leqslant 2m-1. \quad \blacksquare$$

More on consistent asymptotic expansions of a_n and b_n can be found in [27; 40; 43; 4, Chap. 2 (for $m = 2$); 67; 68 (for $f(x) = x^6/6$)] containing the seven first terms of the expansion of $a_n \cdots$

These authors show that these symptotic expansions are actually valid for the solution by using constructions involving partial derivatives of the functions F_n and G_n.

It will indeed appear in the next section that good formal expansions ($F(a', b')$, $G(a', b')$ close to $F(a, b)$, $G(a, b)$) will imply that a'_n and b'_n are themselves close to a_n and b_n and that the link is given by the Fréchet derivative of (F, G).

5. The Fréchet Derivative of (F, G)

It will be found convenient to consider F and G as functions of the variables $2 \log a_k$ and b_k:

DEFINITION 5.1. The Fréchet derivative (Jacobian operator) of (F, G) is the matrix

$$J(a, b)=\begin{bmatrix} \partial F_n(a, b)/\partial \log a_k^2 & \partial F_n(a, b)/\partial b_k \\ \partial G_n(a, b)/\partial \log a_k^2 & \partial G_n(a, b)/\partial b_k \\ k=1, 2,..., & k=0, 1,.... \end{bmatrix} \begin{matrix} ,\quad n=1, 2,..., \\ ,\quad n=0, 1,..., \\ \end{matrix} \tag{5.1}$$

The quadratic form $(J(a, b)\,\rho, \rho)$ associated to

$$\rho=(\sigma_1, \sigma_2, \sigma_3,...; \tau_0, \tau_1, \tau_2,...)$$

is the double sum

$$\begin{aligned}(J(a, b)\,\rho, \rho)= &\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_n \sigma_k\, \partial F_n/\partial \log a_k^2+\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sigma_n \tau_k\, \partial F_n/\partial b_k \\ &+\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \tau_n \sigma_k\, \partial G_n/\partial \log a_k^2+\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \tau_n \tau_k\, \partial G_n/\partial b_k, \end{aligned} \tag{5.2}$$

where $\sigma_1, \sigma_2, \sigma_3,...,$ and $\tau_0, \tau_1, \tau_2,...,$ are sequences of real numbers of which at most finitely many are non zero.

The main result of this section is

THEOREM 5.2. *Let f and w be given by (1.1) and (1.2), respectively. Let $a_n>0$, $n=1, 2, 3,...,$ and b_n, $n=0, 1, 2,...,$ be sequences of real numbers generating orthonormal polynomials p_n, $n=0, 1, 2,...,$ with respect to some nonnegative mass distribution $d\beta$. Then $J(a, b)$ is symmetric, and (5.2) can be written*

$$\begin{aligned}(J(a, b)\,\rho, \rho)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Bigg[&\sum_{n=1}^{\infty} a_n \sigma_n\, p_n(u)\, p_{n-1}(t) \\ &+\sum_{n=0}^{\infty} \tau_n\, p_n(u)\, p_n(t)\Bigg]^2 \frac{f'(u)-f'(t)}{(u-t)}\, d\beta(u)\, d\beta(t). \end{aligned} \tag{5.3}$$

Remark. $J(a, b)$ is positive definite for any admissible sequences $\{a_n, b_n\}$ if *f is convex*. Further, numerical experiments (involving the solution of (1.11) and (1.12) by Newton iteration [27, 38]) suggest the following:

CONJECTURE. If f is given by (1.1), $J(a, b)$ is positive definite at the solution of (1.11) and (1.12).

The proof of the conjecture requires a clever reading of the integral (5.3) when $d\beta(u)\, d\beta(t) = \exp(-f(u) - f(t))\, du\, dt$.

The importance of positive definiteness (or, more generally, bounded invertibility) of J becomes apparent when one wants to compare distances between values of (F, G) and distances between arguments:

$$F_n(a'', b'') - F_n(a', b') = \int_0^1 \left[\sum_{k=1}^{\infty} \sigma_k\, \partial F_n/\partial \log a_k^2 + \sum_{k=0}^{\infty} \tau_k\, \partial F_n/\partial b_k \right] dt,$$

$$G_n(a'', b'') - G_n(a', b') = \int_0^1 \left[\sum_{k=1}^{\infty} \sigma_k\, \partial G_n/\partial \log a_k^2 + \sum_{k=0}^{\infty} \tau_k\, \partial G_n/\partial b_k \right] dt.$$

The integrals are taken on the rectilinear path

$$\begin{aligned} \log a_k^2 &= \log(a_k')^2 + t\sigma_k, \qquad b_k = b_k' + t\tau_k, \quad 0 \leqslant t \leqslant 1; \\ \sigma_k &= \log(a_k'')^2 - \log(a_k')^2 \qquad \text{and} \qquad \tau_k = b_k'' - b_k'. \end{aligned} \tag{5.4}$$

Formally, this gives

$$\sum_{n=1}^{\infty} \sigma_n(F_n'' - F_n') + \sum_{n=0}^{\infty} \tau_n(G_n'' - G_n') = \int_0^1 (J(a, b)\, \rho, \rho)\, dt.$$

If J is positive definite and has a lower bound λ on the path $0 \leqslant t \leqslant 1$, applying Schwarz inequality to the left-hand side,

$$\|(F'', G'') - (F', G')\|\ \|\rho\| \geqslant \lambda \|\rho\|^2$$

which gives an upper bound on $\|\rho\|$, therefore on a_k''/a_k' and $b_k'' - b_k'$. This is basically the method that will be used in the next section in order to establish asymptotic behavior.

Symmetry and positive definiteness of $J(a, b)$ suggests also that the solution of (1.11) and (1.12) is in some way related to a maximization problem, just as it happened for the simplified problem (4.3) (Lemma 4.3).

It should be interesting to investigate the extension of Lew and Quarles method [27] to the present problem.

From (5.1), (1.9), and (1.10), $J(a, b)$ is linear in f, so that, from (1.1),

$$J(a, b) = \sum_{i=0}^{2m-1} (2m - i)\, c_i J_{2m-i-1}(a, b), \tag{5.5}$$

where $J_{2m-i-1}(a, b)$ is the matrix (5.1) constructed with $f'(x) = x^{2m-i-1}$, i.e., with

$$F_n(a, b) = a_n (A^{2m-i-1})_{n,n-1} \qquad \text{and} \qquad G_n(a, b) = (A^{2m-i-1})_{n,n}.$$

It will appear in next section that the behaviour of $J(a, b)$ is in a way dominated by $J_{2m-1}(a, b)$, already known to be positive definite (assuming Theorem 5.2, which will be proved in a moment). A *lower* bound will now be established.

THEOREM 5.3. *Let $J_{2m-1}(a, b)$ be the matrix* (5.1) *corresponding to $f'(x) = x^{2m-1}$, and assume that the real sequences $a = (a_1, a_2, \ldots)$ and $b = (b_0, b_1, b_2, \ldots)$ satisfy*

$$a_n \sim \varphi_n \tag{5.6}$$

and

$$b_n = O(\varphi_n), \tag{5.7}$$

where $\varphi_0, \varphi_1, \ldots,$ are positive and $\varphi_{n+1}/\varphi_n \to 1$ when $n \to \infty$. Then we have for some $C_1 > 0$

$$(J_{2m-1}(a, b)\, \rho, \rho) \geqslant C_1 \sum_{n=0}^{\infty} \{\varphi_n^{2m} \sigma_n^2 + \varphi_n^{2m-2} \tau_n^2\} \tag{5.8}$$

for any $\rho = (\sigma_1, \sigma_2, \ldots; \tau_0, \tau_1, \ldots)$ as in Definition 5.1.

Proof. Using (5.3), orthonormality (Parseval relation) and the inequality

$$(t^{2m-1} - u^{2m-1})/(t-u) \geqslant t^{2m-2}/2, \qquad u, t \in (-\infty, \infty),$$

we see that

$$\begin{aligned}(J(a, b)\, \rho, \rho) &\geqslant \tfrac{1}{2} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} [a_n \sigma'_n p_{n-1}(t) + \tau_n p_n(t)]^2 t^{2m-2} d\beta(t).\\ &= \tfrac{1}{2} \sum_{n=0}^{\infty} \{a_n^2 \sigma_n^2 (A^{2m-2})_{n-1,n-1} + 2a_n \sigma_n \tau_n (A^{2m-2})_{n-1,n}\\ &\quad + \tau_n^2 (A^{2m-2})_{n,n}\}\end{aligned} \tag{5.9}$$

by (1.8) and with the convention $a_0 = 0$. Of course, we assume here that A is the Jacobi matrix associated with $\{a_n\}$ and $\{b_n\}$. Further, we have used the fact that only finitely many σ_n and τ_n are nonzero, to justify all the interchanges.

Now let $A_n^{(2)}$ be the quadratic form

$$\begin{aligned}A_n^{(2)} &= a_n^2 \sigma_n^2 (A^{2m-2})_{n-1,n-1} + 2a_n \sigma_n \tau_n (A^{2m-2})_{n-1,n} + \tau_n^2 (A^{2m-2})_{n,n}\\ &= [a_n \sigma_n \tau_n] \begin{bmatrix} (A^{2m-2})_{n-1,n-1} & (A^{2m-2})_{n-1,n} \\ (A^{2m-2})_{n,n-1} & (A^{2m-2})_{n,n} \end{bmatrix} \begin{bmatrix} a_n \sigma_n \\ \tau_n \end{bmatrix}.\end{aligned} \tag{5.10}$$

Note that for $i, j=0, 1$, $(A^{2m-2})_{n-i,n-j}$ is a sum of products of $2m-2$ elements of the set $\{x_{2n-2m+2}, \ldots, x_{2n+2m-2}\} = \{a_{n-m+1}, \ldots, a_{n+m-1}; b_{n-m+1}, \ldots, b_{n+m-2}\}$ (see Lemma 1.2(iv)). Hence, using the upper bounds (5.6) and (5.7) we see that each element of the matrix in (5.10) is $O(\varphi_n^{2m-2})$. As each eigenvalue of a 2×2 matrix is bounded by twice the largest element (in absolute value), we see that both eigenvalues of this matrix, λ_n and Λ_n say, are $O(\varphi_n^{2m-2})$.

However, we are more interested in lower bounds for the eigenvalues. It is clear from (5.9) that $A_n^{(2)}$ is a nonnegative quadratic form and so λ_n, $\Lambda_n \geqslant 0$. To obtain better lower bounds, we shall find a lower bound for the determinant of the matrix in (5.10). To this end, let B_n be the $2\times\infty$ matrix consisting of the nth and the $(n+1)$th rows of A^{m-1}. As A (and so A^{m-1}) is symmetric, we see that the matrix in (5.10) can be represented as $B_n B_n^T$.

Now we can apply the Cauchy–Binet formula ([13, p. 775; 1, Sect. 36] to evaluate $\det(B_n B_n^T)$. In the following lines $B(i, j \mid k, l)$ denotes the 2×2 matrix formed from the ith and jth rows, and the kth and lth columns of a matrix B. We see that

$$
\begin{aligned}
\det(B_n B_n^T) & \sum_{1\leqslant i_1<i_2<\infty} (\det B_n(1, 2 \mid i_1, i_2))^2 \\
&= \sum_{1\leqslant i_1<i_2<\infty} (\det A^{m-1}(n, n+1 \mid i_1, i_2))^2 \\
&\geqslant \det(A^{m-1}(n, n+1 \mid n+m-1, n+m))^2 \\
&= \{(A^{m-1})_{n-1,n+m-2}(A^{m-1})_{n,n+m-1} \\
&\quad -(A^{m-1})_{n,n+m-2}(A^{m-1})_{n-1,n+m-1}\}^2 \\
&\{(a_n a_{n+1} \cdots a_{n+m-2})(a_{n+1} a_{n+2} \cdots a_{n+m-1})\}^2
\end{aligned}
$$

from Lemma 1.2(i) and (ii). We obtain

$$\lambda_n \Lambda_n = \det(B_n B_n^T) \geqslant C\varphi_n^{4m-4},$$

by the lower bound (5.6). As λ_n, Λ_n are $O(\varphi_n^{2m-2})$, we have

$$\lambda_n \sim \Lambda_n \sim \varphi_n^{2m-2}.$$

We deduce that

$$A_n^{(2)} \geqslant C\varphi_n^{2m-2}((a_n\sigma_n)^2 + \tau_n^2),$$

and then (5.6) and (5.9) yield the result. ∎

Proof of Theorem 5.2. We have already introduced in (5.5) the matrices $J_k(a, b)$ corresponding to $f'(x)=x^k$. Let

$$J(a, b; z)=\sum_{k=0}^{\infty} z^{-k-1}J_k(a, b)$$

be their generating function. An expression of $J(a, b; z)$ will be established, from which the coefficients $J_k(a, b)$ will be extracted, and (5.5) will give (5.3).

As

$$(z-x)^{-1}=\sum_{k=0}^{\infty} z^{-k-1}x^k$$

and

$$(zI-A)^{-1}=\sum_{k=0}^{\infty} z^{-k-1}A^k,$$

the matrix $J(a, b; z)$ is obtained by putting

$$F_n(a, b)=a_n(zI-A)^{-1}_{n,n-1} \qquad \text{and} \qquad G_n(a, b)=(zI-A)^{-1}_{n,n}$$

in (5.1):

$$J(a, b; z)=\begin{bmatrix} \dfrac{\partial}{\partial \log a_k^2}\{a_n(zI-A)^{-1}_{n,n-1}\} & \dfrac{\partial}{\partial b_k}\{a_n(zI-A)^{-1}_{n,n-1}\} \\ \dfrac{\partial}{\partial \log a_k^2}(zI-A)^{-1}_{n,n} & \dfrac{\partial}{\partial b_k}(zI-A)^{-1}_{n,n} \end{bmatrix}.$$

First, note that if x is some parameter occuring in A, then a straightforward calculation shows that

$$\partial(zI-A)^{-1}/\partial x=(zI-A)^{-1}\,\partial A/\partial x(zI-A)^{-1}.$$

For $x=\log a_k^2$, $\partial A/\partial x$ contains only two nonzero elements, namely $(\partial A/\partial x)_{k,k-1}$ and $(\partial A/\partial x)_{k-1,k}$, while for $x=b_k$, the only nonzero element of $\partial A/\partial x$ is $(\partial A/\partial x)_{k,k}$. Note too that $\partial a_n/\partial \log a_k^2=\delta_{nk}a_n/2$, $k, n=1, 2, 3,\ldots$.

These operations on $(zI-A)^{-1}$ may be considered as formal ones, as a mere summary of operations performed separately on each power A^k, but $(zI-A)^{-1}$ is perfectly well defined for nonreal z if the moment problem is determinate (cf. [80, Sects. 60 and 61]; see also Sect. 7 of the present paper).

In view of (1.9), (1.10), and (5.1),

$$J(a,b,z) = \begin{bmatrix} \frac{a_n a_k}{2}[(zI-A)^{-1}_{n,k-1}(zI-A)^{-1}_{k,n-1}+(zI-A)^{-1}_{n,k}(zI-A)^{-1}_{k-1,n-1}] \\ \quad +(a_n/2)(zI-A)^{-1}_{n,n-1}\delta_{n,k} & a_n(zI-A)^{-1}_{n,k}(zI-A)^{-1}_{k,n-1} \\ (a_k/2)[(zI-A)^{-1}_{n,k-1}(zI-A)^{-1}_{k,n}+(zI-A)^{-1}_{n,k}(zI-A)^{-1}_{k-1,n}] & (zI-A)^{-1}_{n,k}(zI-A)^{-1}_{k,n} \end{bmatrix}. \tag{5.11}$$

Since A and $(zI-A)^{-1}$ are symmetric, it is clear from (5.11) that the diagonal blocks in $J(a, b)$ are symmetric. Further we see that the bottom left-hand element in (5.11) may be replaced by $a_k[(zI-A)^{-1}_{n,k}(zI-A)^{-1}_{n,k-1}]$ and hence the off-diagonal blocks in $J(a, b)$ are the transposes of each another. Thus $J(a, b)$ is symmetric.

The second part of Theorem 5.2 is established using the spectral representation [2, Chap.4; 6, Chap. 7, Sect.1, formula (1.37)].

$$(zI-A)^{-1}_{n,k} = \int_{-\infty}^{\infty} (z-u)^{-1} p_n(u)\, p_k(u)\, d\beta(u) \tag{5.12}$$

which extends (1.8). First, we must clean up the writing of the elements of the upper left-hand block in (5.11). To do this, we use [80, formula (60.3) and Theorem 61.1],

$$(zI-A)^{-1}_{n,k} = \begin{cases} p_n(z)\, q_k(z) & \text{if } n \leqslant k \\ p_k(z)\, q_n(z) & \text{if } k \leqslant n, \end{cases}$$

where $q_n(z) = \int_{-\infty}^{\infty} (z-x)^{-1} p_n(x)\, d\beta(x)$, z nonreal (one may use (5.12), expressing orthogonality of $p_n(x)$ and $(p_k(z)-p_k(x))/(z-x)$ for $k \leqslant n$). The (n, k) element of the upper left block is then

$$\frac{a_n a_k}{2}(p_n q_{k-1} q_k p_{n-1} + p_n q_k q_{k-1} p_{n-1}) = a_n a_k p_n q_k q_{k-1} p_{n-1} \qquad \text{if } n<k$$

$$\frac{a_n a_k}{2}(q_n p_{k-1} p_k q_{n-1} + q_n p_k p_{k-1} q_{n-1}) = a_n a_k q_n p_k p_{k-1} q_{n-1} \qquad \text{if } n>k$$

$$\frac{a_n^2}{2}(q_n^2 p_{n-1}^2 + p_n q_n p_{n-1} q_{n-1}) + a_n q_n p_{n-1}/2 \qquad \text{if } n=k.$$

This latter expression turns into $a_n^2 p_n q_n p_{n-1} q_{n-1}$, as $q_n p_{n-1} - q_{n-1} p_n = -1/a_n$ (determinant formula in Wall [80, Sect. 60]).

In any case, the element is $(zI-A)^{-1}_{n,k}(zI-A)^{-1}_{k-1,n-1}$,

$$J(a,b,z)=\begin{bmatrix} a_n a_k (zI-A)^{-1}_{n,k}(zI-A)^{-1}_{k-1,n-1} & a_n (zI-A)^{-1}_{n,k}(zI-A)^{-1}_{k,n-1} \\ a_k (zI-A)^{-1}_{n,k}(zI-A)^{-1}_{k-1,n} & (zI-A)^{-1}_{n,k}(zI-A)^{-1}_{k,n} \end{bmatrix}.$$

Now, from (5.12)

$$J(a,b,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(z-u)^{-1}(z-t)^{-1}$$
$$\times\begin{bmatrix} a_n a_k p_n(u)\,p_k(u)\,p_{k-1}(t)\,p_{n-1}(t) & a_n p_n(u)\,p_k(u)\,p_k(t)\,p_{n-1}(t) \\ a_k p_n(u)\,p_k(u)\,p_{k-1}(t)\,p_n(t) & p_n(u)\,p_k(u)\,p_k(t)\,p_n(t) \end{bmatrix}$$
$$\times\, d\beta(u)\,d\beta(t).$$

Finally, we may rewrite this last matrix as a power series in z^{-1} using the identity

$$(z-u)^{-1}(z-t)^{-1}=(u-t)^{-1}[(z-u)^{-1}-(z-t)^{-1}]$$
$$=\sum_{k=0}^{\infty}(u-t)^{-1}(u^k-t^k)\,z^{-k-1}.$$

showing

$$J_k=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(u^k-t^k)(u-t)^{-1}$$
$$\times\begin{bmatrix} a_n a_k p_n(u)\,p_k(u)\,p_{k-1}(t)\,p_{n-1}(t) & a_n p_n(u)\,p_k(u)\,p_k(t)\,p_{n-1}(t) \\ a_k p_n(u)\,p_k(u)\,p_{k-1}(t)\,p_n(t) & p_n(u)\,p_k(u)\,p_k(t)\,p_n(t) \end{bmatrix}$$
$$\times\, d\beta(u)\,d\beta(t).$$

And (5.5) gives (5.3). ∎

6. Asymptotic Behaviour of the Solution

In this section, we shall prove the main result of this paper which solves a generalization of Freud's Conjecture [16–18]:

Theorem 6.1. *Let f and w be given by* (1.1) *and* (1.2), *respectively. Let a_n, $n=1, 2,...$, and b_n, $n=0, 1, 2,...$, be the admissible solution of* (1.11) *and* (1.12). *Let $C(2m)$ be given by* (4.2). *Then*

$$\lim_{n \to \infty} \frac{a_n}{\{n/(c_0 C(2m))\}^{1/(2m)}} = 1, \tag{6.1}$$

$$b_n = \circ(n^{1/(2m)}), \qquad n \to \infty. \tag{6.2}$$

Proof. First, let us consider new variables

$$\bar{a}_n = n^{-1/(2m)} a_n, \qquad \bar{b}_n = n^{-1(2m)} b_n, \quad n \geq 1. \tag{6.3}$$

This transformation, which can be considered as a *nonuniform* scaling, has been introduced in [56]. It allows significant simplification of asymptotic behavior determination (cf. [56], where $\bar{a}_n^2$ is called A_n, and [38]). One has of course to prove that $\{\bar{a}_n\}$ and $\{\bar{b}_n\}$ have the limits $[c_o C(2m)]^{-1/(2m)}$ and 0 when $n \to \infty$. The only sequences that we have to consider satisfy (see (3.1) and (3.2))

$$\bar{a}_n \sim 1 \qquad \text{and} \qquad \bar{b}_n = O(1). \tag{6.4}$$

Now, we discuss the writing of F_n and G_n in terms of $\bar{a}$ and $\bar{b}$: each term $c_i x^{2m-i}$ occurring in the polynomial $f(x)$ generates homogeneous polynomials in $a_{n-m+1}, \ldots, a_{n+m-1}$, $b_{n-m+1}, \ldots, b_{n+m-1}$ (see Lemma 1.3). The total degree of these polynomials in F_n and G_n are, respectively, $2m-i$ and $2m-i-1$. Let us write (1.19) and (1.20) in the form

$$\begin{aligned} F_n(a, b) &= \sum_{i=0}^{2m-2} c_i F_n^{(2m-i)}(a, b), \\ G_n(a, b) &= \sum_{i=0}^{2m-1} c_i G_n^{(2m-i)}(a, b). \end{aligned} \tag{6.5}$$

Turning to the $\bar{a}$'s and $\bar{b}$'s, each term of these polynomials exhibits now a product of factors $(n+j)^{1/(2m)}$, with $j \in$ some set $J \subset [1-m, m-1]$

$$\prod_{j \in J} x_{n+j} = \prod_{j \in J} (n+j)^{1/(2m)} \prod_{j \in J} \bar{x}_{n+j} = n^{k/(2m)} \prod_{j \in J} \bar{x}_{n+j} + O(n^{k/(2m)-1}),$$

where the x's are a's or b's and k is the degree. The 0 term is of course justified by (6.4). This gives

$$\begin{aligned} F_n(a, b) &= \sum_{i=0}^{2m-2} n^{1-i/(2m)} c_i F_n^{(2m-i)}(\bar{a}, \bar{b}) + O(1), \\ G_n(a, b) &= \sum_{i=0}^{2m-1} n^{(2m-i-1)/(2m)} c_i G_n^{(2m-i)}(\bar{a}, \bar{b}) + O(n^{-1/(2m)}). \end{aligned} \tag{6.6}$$

We are now able to discuss the closeness of $\{a'_n\}$and $\{a''_n\}$, $\{b'_n\}$ and $\{b''_n\}$ when (F'', G'') and (F', G') are close together. To this end, let $\{a''_n\}$

and $\{b_n''\}$ denote the admissible solution of (1.11) and (1.12). Further, let $\{a_n'\}$ and $\{b_n'\}$ denote the consistent approximate solution with the properties listed in Theorem 4.2. The upper and lower bounds in (3.1) and (3.2), and therefore (6.4), are valid uniformly on the rectilinear path defined by (5.4). Remark that $\bar{\sigma}_k = \sigma_k$, $\bar{\tau}_k = k^{-1/(2m)}\tau_k$. From (4.10), (4.11), and (6.6),

$$\sum_{i=0}^{2m-2} n^{-i(2m)} c_i [F_n^{(2m-i)}(\bar{a}'', \bar{b}'') - F_n^{(2m-i)}(\bar{a}', \bar{b}')] = O(1/n),$$

$$\sum_{i=0}^{2m-1} n^{-i(2m)} c_i [G_n^{(2m-i)}(\bar{a}'', \bar{b}'') - G_n^{(2m-i)}(\bar{a}', \bar{b}')] = O(1/n).$$

Each of these differences is now written as an integral on the path (5.4):

$$\begin{aligned} &\sum_{i=0}^{2m-2} n^{-i/(2m)} c_i \int_0^1 \Bigg[\sum_k \bar{\sigma}_k \, \partial F_n^{(2m-i)}/\partial \log \bar{a}_k^2 \\ &\qquad + \sum_k \bar{\tau}_k \, \partial F_n^{(2m-i)}/\partial \bar{b}_k \Bigg] dt = O(1/n), \\ &\sum_{i=0}^{2m-1} n^{-i/(2m)} c_i \int_0^1 \Bigg[\sum_k \bar{\sigma}_k \, \partial G_n^{(2m-i)}/\partial \log \bar{a}_k^2 \\ &\qquad + \sum_k \bar{\tau}_k \, \partial G_n^{(2m-i)}/\partial \bar{b}_k \Bigg] dt = O(1/n), \end{aligned} \tag{6.7}$$

where the interior sums involve $k = n - m + 1, \ldots, n + m - 1$. Now, we build bilinear forms by multiplying each of these relations by $\bar{\sigma}_n$ or $\bar{\tau}_n$, for $v \leqslant n \leqslant N$. Let

$$\bar{\rho} = (0, \ldots, 0, \bar{\sigma}_v, \ldots, \bar{\sigma}_N, 0, \ldots; 0, \ldots, 0, \bar{\tau}_v, \ldots, \bar{\tau}_N, 0, \ldots),$$

$$\begin{aligned} \bar{\rho}^* = (&0, \ldots, 0, \bar{\sigma}_{v-m+1}, \ldots, \bar{\sigma}_{N+m-1}, 0, \ldots; \\ &0, \ldots, 0, \bar{\tau}_{v-m+1}, \ldots, \bar{\tau}_{N+m-1}, 0, \ldots). \end{aligned}$$

$$\begin{aligned} \bar{\rho}^{(i)} = (&0, \ldots, 0, v^{-i/(2m)}\bar{\sigma}_v, \ldots, N^{-i/(2m)}\bar{\sigma}_N, 0, \ldots; \\ &0, \ldots, 0, v^{-i/(2m)}\bar{\tau}_v, \ldots, N^{-i/(2m)}\bar{\tau}_N, 0, \ldots) \end{aligned}$$

We have, from (5.5)

$$\sum_{i=0}^{2m-1} (2m-i)\, c_i \int_0^1 (J_{2m-i-1}(\bar{a}, \bar{b})\, \bar{\rho}^*, \bar{\rho}^{(i)})\, dt = \sum_v^N \varepsilon_n \bar{\sigma}_n + \eta_n \bar{\tau}_n,$$

where ε_n and η_n are the $O(1/n)$ terms of the right-hand side of (6.7). Let

$$\|\bar{\rho}\|^2 = \sum_{n=v}^{N} \bar{\sigma}_n^2 + \bar{\tau}_n^2, \qquad \|\tilde{\rho}\|^2 = \sum_{v-m+1}^{v+m-1} \bar{\sigma}_n^2 + \bar{\tau}_n^2 + \sum_{N-m+1}^{N+m-1} \bar{\sigma}_n^2 + \bar{\tau}_n^2.$$

The bilinear forms $(J_{2m-i-1}(\bar{a}, \bar{b})\,\bar{\rho}^*, \bar{\rho}^{(i)})$ are easily bounded by const. $\nu^{-i/(2m)}(\|\tilde{\rho}\|^2 + \|\bar{\rho}\|^2)$, as the $\bar{a}$'s, $\bar{b}$'s, $\bar{\sigma}$'s, and $\bar{\tau}$'s are themselves bounded and the bilinear forms are made of a finite number of sums of the form

$$\sum_{n=\nu}^{N} n^{-i/(2m)} C_n \bar{\sigma}_n \bar{\sigma}_{n+j} \leqslant \nu^{-i/(2m)} (\max_n |C_n|) \sum_{\nu-|j|}^{\nu+|j|} \bar{\sigma}_n^2,$$

$$\sum_{n=\nu}^{N} n^{-i/(2m)} D_n \bar{\sigma}_n \bar{\tau}_{n+j} \leqslant \nu^{-i/(2)m} (\max_n |D_n|) \frac{1}{2} \sum_{\nu-|j|}^{\nu+|j|} \bar{\sigma}_n^2 + \bar{\tau}_n^2, \quad \text{etc.}$$

We have also a *lower bound* for $(J_{2m-1}(\bar{a}, \bar{b})\,\bar{\rho}^*, \bar{\rho})$ giving $(J_{2m-1}(\bar{a}, \bar{b})\,\bar{\rho}, \bar{\rho})$ and a finite number of terms involving $\bar{\sigma}_n$ and $\bar{\tau}_n$, $n = \nu - m + 1, \ldots, \nu + m - 1;\ N - m + 1, \ldots, N + m - 1$ gathered in $\|\tilde{\rho}\|^2$. Indeed, Theorem 5.3 holds, with plain constants for φ_n, uniformly on the path (5.4),

$$\begin{aligned}(2m-1)\, c_0 \int_0^1 &(J_{2m-1}(\bar{a}, \bar{b})\,\bar{\rho}^*, \bar{\rho})\, dt \\ &\geqslant (2m-1)\, c_0 \int_0^1 (J_{2m-1}(\bar{a}, \bar{b})\,\bar{\rho}, \bar{\rho})\, dt - C\|\tilde{\rho}\|^2 \\ &\geqslant D\|\bar{\rho}\|^2 - C\|\tilde{\rho}\|^2.\end{aligned}$$

The result is

$$D\|\bar{\rho}\|^2 - C\|\tilde{\rho}\|^2 - C'\nu^{-1/(2m)}(\|\tilde{\rho}\|^2 + \|\bar{\rho}\|^2) \leqslant \left| \sum_{\nu}^{N} \varepsilon_n \bar{\sigma}_n + \eta_n \bar{\tau}_n \right|;$$

Cauchy–Schwarz's inequality for the right-hand side yields

$$D\|\bar{\rho}\|^2 - C\|\tilde{\rho}\|^2 - C'\nu^{-1/(2m)}(\|\tilde{\rho}\|^2 + \|\bar{\rho}\|^2) \leqslant E\nu^{-1/2}\|\bar{\rho}\|.$$

To get a useful upper bound for $\|\bar{\rho}\|$, we just have to take ν large enough to have $D - C'\nu^{-1/(2m)} \geqslant D' > 0$,

$$D'\|\bar{\rho}\|^2 - C''\|\tilde{\rho}\|^2 \leqslant E\nu^{-1/2}\|\bar{\rho}\|. \tag{6.8}$$

As $\bar{\sigma}_n$, $\bar{\tau}_n$, and therefore $\|\tilde{\rho}\|$ are $O(1)$, we find $\|\bar{\rho}\| \leqslant \text{const.} < \infty$, independently of N, so that $\{\bar{\sigma}_n\}$ and $\{\bar{\tau}_n\}$ are square summable sequences. In particular,

$$\bar{\sigma}_n = \log(a_n'')^2 - \log(a_n')^2 \to 0, \qquad \bar{\tau}_n = n^{-1/(2m)}(b_n'' - b_n') \to 0, \quad n \to \infty. \quad \blacksquare$$

Remark. With a little supplementary effort, one can deduce from (6.8)

$$\bar{\sigma}_n \quad \text{and} \quad \bar{\tau}_n = O(n^{-1/2}),$$

i.e., the admissible solution a_n, b_n of (1.11) and (1.12) and the consistent approximate solution a'_n, b'_n of Theorem 4.2 are related by

$$a_n - a'_n \quad \text{and} \quad b_n - b'_n = O(n^{-1/2+1/(2m)}). \tag{6.9}$$

Indeed, let us take $N = \infty$ and denote

$$\|\bar{\rho}_\nu\|^2 = \sum_{\nu}^{\infty} \bar{\sigma}_n^2 + \bar{\tau}_n^2$$

Then, $\|\tilde{\rho}\|^2 = \|\bar{\rho}_{\nu-m+1}\|^2 - \|\bar{\rho}_{\nu+m}\|^2$, and (6.8) becomes

$$D'\|\bar{\rho}_\nu\|^2 - C''(\|\bar{\rho}_{\nu-m+1}\|^2 - \|\bar{\rho}_{\nu+m}\|^2) \leqslant E\nu^{-1/2}\|\bar{\rho}_\nu\|$$

that can be widened in

$$\begin{aligned} D'\|\bar{\rho}_{\nu+m}\|^2 - C''(\|\bar{\rho}_{\nu-m+1}\|^2 - \|\bar{\rho}_{\nu+m}\|^2) \\ \leqslant E(\nu - m + 1)^{-1/2}\|\bar{\rho}_{\nu-m+1}\|. \end{aligned}$$

Multiplying by $\nu - m + 1$ and introducing $D'' > 0$ such that $(D' + C'')(\nu - m + 1) \geqslant (D'' + C'')(\nu + m)$ shows that

$$(D'' + C'')\, X_{\nu+m} \leqslant E(X_{\nu-m+1})^{1/2} + C''X_{\nu-m+1}:$$

$X_\nu = \nu\|\bar{\rho}_\nu\|^2$ must remain bounded when ν increases. ∎

The methods developed in [27, 40, 43] (cf. [4, 5, 67]) allow the construction of asymptotic expansions superseding the scope of this remark.

7. Nonpolynomial Exponents; Towards the Freud's Conjecture

How to define the Freud's equations (1.11) and (1.12) when f is not a polynomial? As already encountered, $f'(A)$ can be defined from the spectral form $(f'(A))_{n,m} = \int_{-\infty}^{\infty} p_n(t)\, p_m(t)\, f'(t)\, d\beta(t)$, but this brings us back to the starting point! What matters is that $f'(A)$ depends actually on the two sequences $\{a_n\}$ and $\{b_n\}$. This will be the case if the corresponding Hamburger moment problem is determinate (and A is selfadjoint and the spectral theorem holds [2, 6]).

A "hard" (if what precedes may be called a "soft" definition) construction of $F_n(a, b)$ and $G_n(a, b)$ is to work with a sequence of polynomial approximations of f. Convergence asks for completeness of polynomials in the appropriate L_2 space, but this is also related to the determinateness of the moment problem [2].

For $f(x)=|x|^\alpha$ it seems therefore that only $\alpha \geqslant 1$ can be treated by the present method. Lemma 7.5 of [30] confirms that polynomials are not dense in the required L_p spaces if $\alpha<1$ since $\exp(-|x|^\alpha)$ is not an extremal solution of its moment problem (cf. [60, Corollary 7] where the above result is generalized).

Here is a demonstration of the polynomial approximation technique:

THEOREM 7.1. *For* $f(x)=|x|^\alpha$, $\alpha>1$, $a_n=(n/C(\alpha))^{1/\alpha}$, $b_n=0$ *is a consistent approximate solution of* (1.11) *and* (1.12).

Proof. We start with the Chebyshev series identity

$$f'(x)=\alpha|x|^{\alpha-1}\operatorname{sign} x=\sum_0^\infty \beta_k T_{2k+1}(x), \qquad -1\leqslant x\leqslant 1$$

$$=\zeta^{\alpha-1}\sum_0^\infty \beta_k T_{2k+1}(x/\zeta), \qquad -\zeta\leqslant x\leqslant \zeta$$

$$\beta_k=2\alpha\pi^{-1/2}(-1)^k\Gamma((\alpha+1/2)\,\Gamma(k+1-\alpha/2)/[\Gamma(1-\alpha/2)\,\Gamma(k+1+\alpha/2)]$$

$$\sim \text{const.}(-1)^k k^{-\alpha} \qquad \text{for large } k.$$

If we stop with T_{2N+1}, we obtain a polynomial $\sum_0^N c_{k,N}\zeta^{\alpha-2k-2}x^{2k+1}$ and an error $=O((\zeta/N)^{\alpha-1})$, $-\zeta\leqslant x\leqslant\zeta$. For a *fixed* n, and with this polynomial, $F_n(a)$ is approximated by $a_n\sum_0^N c_{k,N}\zeta^{\alpha-2k-2}(A^{2k+1})_{n,n-1}$, involving $a_{n-2N},\ldots, a_{n+2N}$ approximately all equal to a_n (if N is much smaller than n). Then, $(A^{2k+1})_{n,n-1}\sim\binom{2k+1}{k}a_n^{2k+1}=\pi^{-1}2^{2k+1}\int_0^1 t^{k+1/2}(1-t)^{-1/2}dt\,a_n^{2k+1}$,

$$F_n(a)\sim a_n\pi^{-1}\int_0^1 \alpha(2t^{1/2}a_n)^{\alpha-1}(1-t)^{-1/2}dt$$

$$=C(\alpha)\,a_n^\alpha \qquad (2a_n<\zeta\ll N\ll n). \quad \blacksquare$$

8. CONCLUSIONS

Further investigations can be considered, according to the sections of the present work:

Section 1: Definition of Freud's equations, as just discussed, is linked to determinateness of the moment problem.

Section 2: Unicity, seems to be rather easy to extend to nonpolynomial functions f, the case $f(x)\sim|x|^{1+\eta}$, $\eta>0$, has already been discussed (Remark 2.4). One may even consider only $f'(x)\sim\operatorname{sign}(x)\,\phi(x)$, when $\phi(x)$ is ultimately increasing and $\to\infty$ when $x\to\pm\infty$.

Section 3: Bounds, the job is done [9, 10, 23, 30, 31, 49, 55].

Section 4: Design of good consistent approximate solution, still requires technical developments. This will perhaps be the most difficult task. Remark that Theorem 7.1 does not supply order terms for the error $F_n(a) - n$. Smooth function, such as $f(x) = (x^2 + 1)^{\alpha/2}$ instead of $|x|^\alpha$ should make things simpler.

Section 5: Fréchet Derivative, seems amenable to extension, through carefully designed conditions for f, and a theorem of derivability. As formula (5.3) has a meaning for non polynomial functions f, one knows what must be proved.

Section 6: Asymptotics of recurrence coefficients, should welcome more powerful methods, as there would be less constraints on the consistent approximate solution needed in the final proof. In this section, as in Section 4, the fact that A and J are band matrices has been repeatedly used. This seems to be the main weakness of the present approach.

The history of Freud's Conjectures and their impact on recent developments in the theory of orthogonal polynomials on infinite intervals is discussed in great detail in [57].

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When he learned that I was working on Freud's Conjecture, P. Nevai expressed a very strong wish that I submit a paper for these special volumes of the Journal. We exchanged letters, where he carefully delineated the subject matter from my loose suggestions. He also elaborated on the long references list. So he is the first to thank for this paper. When I finally submitted to him the result of my efforts, he asked Doron S. Lubinsky for his advice. In response, Doron wrote the paper in its present form. He supplied the missing links, cleared cumbersome pieces of reasoning and introduced at some places completely new material (especially in Section 2).

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